Calculus II, Math 3495
Abedi

i) **Convergence of a Geometric Series**

A geometric series $\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \ldots.$ with ratio $r$ diverges if $|r| \geq 1$ and converges if $|r| < 1$.

$$\sum_{n=0}^{\infty} ar^n = \begin{cases} \text{Converges to } \dfrac{a}{1-r} & \text{if } -1<r<1 \\ \text{Diverges} & \text{if } |r| \geq 1\ (r \leq -1 \vee r \geq 1) \end{cases}$$

$r \leq -1$ or $r \geq 1$

ii) **Divergent Test (*n*th-Term Test for Divergence)**

If the sequence $\{a_n\}$ does not converge to 0, then the series $\sum a_n$ diverges.

( ie.; If $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum a_n$ diverges)

iii) **Integral Test**

If $f$ is positive, continuous, and decreasing for $x \geq 1$ and $f(n) = a_n, n \geq 0$ ,then $\sum_{n=1}^{\infty} a_n$ & $\int_1^{\infty} f(x)dx$

either both converge or both diverge.

iv) **Convergence of *p*-series**

The $p$-series $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots\cdots \begin{cases} \text{converges if } p > 1, \text{ and} \\ \text{diverges if } p \leq 1 \end{cases}$

Examples: $\sum_{n=1}^{\infty} \frac{1}{n^2}$ Converges, : $\sum_{n=1}^{\infty} \frac{1}{n}$ (Harmonic Series) Diverges

Note:: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \cdots\cdots$ (Alternating Harmonic Series) Converges

v) **The Alternating Series Test**

The alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} a_n = a_1 - a_2 + a_3 - a_4 \ldots\ldots\ldots$ $a_n > 0$
satisfies

a) $a_{n+1} \leq a_n$ for all $n$ (i.e; the sequence $\{a_n\}_{n=1}^{\infty}$ is decreasing)

b) $\lim_{n\to\infty} a_n = 0$

then the series is convergent.

vi) **Direct Comparison Test**

Let $0 \le a_n \le b_n$ for all $n$.

a) If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

b) If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

vii) **Limit Comparison Test**

Suppose $a_n > 0$, $b_n > 0$, and $\lim_{n\to\infty} \frac{a_n}{b_n} = L$

where $L$ is **finite** and **positive**. Then the two series $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

vii) **The Ratio Test**

a) If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \rho < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

b) If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \rho > 1$ or $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Note that if $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$, the Ratio Test gives no information.