Calculus I, Math 2255

Let $f(x) = \sqrt{x^2 - 2x} - 1$.

i) Find $f'(3)$.

$$f'(3) = \lim_{a\to 3} \frac{f(a) - f(3)}{a - 3}$$
$$= \lim_{a\to 3} \frac{\sqrt{a^2 - 2a} - 1 - (\sqrt{3} - 1)}{a - 2}$$
$$= \lim_{a\to 3} \frac{\sqrt{a^2 - 2a} - (\sqrt{3})}{a - 3} \cdot \frac{\sqrt{a^2 - 2a} + (\sqrt{3})}{\sqrt{a^2 - 2a} + (\sqrt{3})}$$
$$= \lim_{a\to 3} \frac{a^2 - 2a - (\sqrt{3})^2}{(a-3)\left[\sqrt{a^2 - 2a} + (\sqrt{3} - 1)\right]} = \lim_{a\to 3} \frac{a^2 - 2a - 3}{(a-3)\left[\sqrt{a^2 - 2a} + (\sqrt{3})\right]} =$$

$$\lim_{a\to 3} \frac{(a-3)(a+1)}{(a-3)\left[\sqrt{a^2-2a} + (\sqrt{3})\right]} = \lim_{a\to 3} \frac{a+1}{\left[\sqrt{a^2-2a} + (\sqrt{3})\right]} = \frac{4}{2\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

ii) Find $f'(x)$.

$$f'(x) = \lim_{a\to x} \frac{f(a) - f(x)}{a - x}$$
$$= \lim_{a\to x} \frac{\sqrt{a^2 - 2a} - 1 - (\sqrt{x^2 - 2x} - 1)}{a - x}$$
$$= \lim_{a\to x} \frac{\sqrt{a^2 - 2a} - (\sqrt{x^2 - 2x})}{a - x} \cdot \frac{\sqrt{a^2 - 2a} + (\sqrt{x^2 - 2x})}{\sqrt{a^2 - 2a} + (\sqrt{x^2 - 2x})}$$
$$= \lim_{a\to x} \frac{a^2 - 2a - (x^2 - 2x)}{(a-x)\left[\sqrt{a^2 - 2a} + (\sqrt{x^2 - 2x})\right]}$$
$$= \lim_{a\to x} \frac{a^2 - x^2 + 2x - 2a}{(a-x)\left[\sqrt{a^2 - 2a} + \sqrt{x^2 - 2x}\right]} =$$

$$\lim_{a\to x} \frac{(a-x)(a+x) - 2(a-x)}{(a-x)\left[\sqrt{a^2 - 2a} + \sqrt{x^2 - 2x}\right]} = \lim_{a\to x} \frac{(a-x)(a+x-2)}{(a-x)\left[\sqrt{a^2 - 2a} + \sqrt{x^2 - 2x}\right]}$$
$$= \lim_{a\to x} \frac{a + x - 2}{\left[\sqrt{a^2 - 2a} + \sqrt{x^2 - 2x}\right]} = \frac{2x - 2}{2\sqrt{x^2 - 2x}} = \frac{x - 1}{\sqrt{x^2 - 2x}}$$

Note that $f'(3) = \frac{(3)-1}{\sqrt{3^2 - 2(3)}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$

iii) Find $f'(x)$ using the other definiation. $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{\sqrt{(x+h)^2-2(x+h)}-1-\left(\sqrt{x^2-2x}\ -1\right)}{h}$$

$$= \lim_{h\to 0}\frac{\sqrt{(x+h)^2-2(x+h)}-\left(\sqrt{x^2-2x}\ \right)}{h}\cdot\frac{\sqrt{(x+h)^2-2(x+h)}+\left(\sqrt{x^2-2x}\ \right)}{\sqrt{(x+h)^2-2(x+h)}+\left(\sqrt{x^2-2x}\ \right)}$$

$$= \lim_{h\to 0}\frac{(x+h)^2-2(x+h)-(x^2-2x)}{h\left[\sqrt{(x+h)^2-2(x+h)}+\left(\sqrt{x^2-2x}\right)\right]}$$

$$= \lim_{h\to 0}\frac{x^2+2hx+h^2-2x-2h-x^2+2x}{h\left[\sqrt{(x+h)^2-2(x+h)}+\sqrt{x^2-2x}\right]} = \lim_{h\to 0}\frac{2hx+h^2-2h}{h\left[\sqrt{(x+h)^2-2(x+h)}+\sqrt{x^2-2x}\right]}$$

$$= \lim_{h\to 0}\frac{h(2x+h-2)}{h\left[\sqrt{(x+h)^2-2(x+h)}+\sqrt{x^2-2x}\right]} = \lim_{h\to 0}\frac{2x+h-2}{\sqrt{(x+h)^2-2(x+h)}+\sqrt{x^2-2x}}$$

$$= \frac{2x-2}{2\sqrt{x^2-2x}} = \frac{x-1}{\sqrt{x^2-2x}}$$