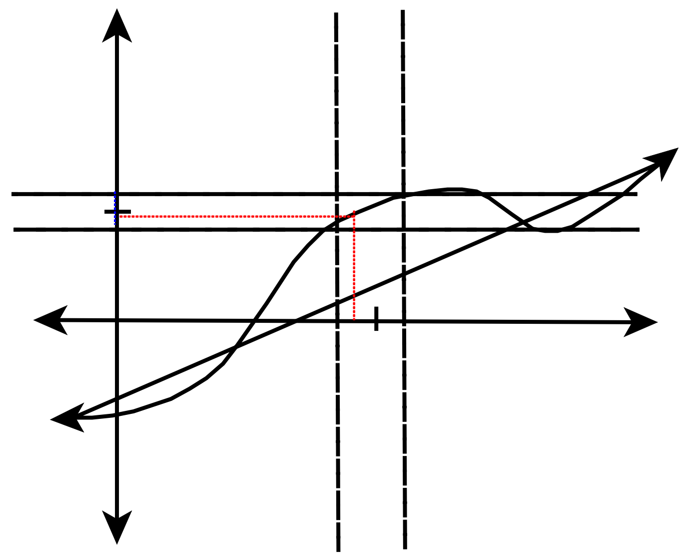
Review Sheet
Calculus I

If $f(x)$ is defined on an open interval containing $x = c$ except possibly at c and L is a real number then $\lim_{x \to c} f(x) = L$ means that $f(x)$ can be arbitrary close to L by taking x sufficiently close to c. This is amount to say that given any $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $|x - c| < \delta$.



Exercise;

Use the definition to prove that $L = \lim_{x \to -3} (2x + 5)$ exists.

Properties of limits:

Let $b, c \in \Re$, and let n be a positive integer. Let f and g be functions such that $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = K$. Then

1. $\lim_{x \to c} [bf(x)] = bL$

2. $\lim_{x \to c} [f(x) \pm g(x)] = L \pm K$

3. $\lim_{x \to c} [f(x) \cdot g(x)] = L \cdot K$

4. $\lim_{x\to c}\left[\frac{f(x)}{g(x)}\right] = \frac{L}{K}, \quad K \neq 0$

5. $\lim_{x\to c}\left[f(x)\right]^n = L^n$

$\lim_{x\to c} f(x)$ may be determined for the following functions by evaluating $f(c)$:

1. Polynomials
2. Trigonometric functions when $f(c)$ is defined
3. Rational functions when $f(c)$ is defined.

Remember that when $f(c) = g(c)$ for all x in an open interval containing c except possibly at $x = c$, and if $\lim_{x\to c} g(x) = L$ then $\lim_{x\to c} f(x) = L$. This allows us to simplify rational functions and to rationalize the numerator.

Squeezing Theorem:

If $h(x) \leq f(x) \leq g(x)$, for all x in an open interval containing c except possibly at $x = c$ and $\lim_{x\to c} h(x) = \lim_{x\to c} g(x) = L$ then $\lim_{x\to c} f(x) = L$. We used this to prove the following:

1. $\lim_{x\to 0}\frac{\sin x}{x} = 1$

2. $\lim_{x\to 0}\frac{1 - \cos x}{x} = 0$.

One sided limits:

We write $\lim_{x\to c^+} f(x) = L$ and say the limit of $f(x)$ as x approaches to c from the right (or right hand limit of $f(x)$ as x approaches to c) is equal to L if we can make the values of $f(x)$ arbitrary close to L by taking x to be sufficiently close to c and x greater than c.

Similarly we write $\lim_{x \to c^-} f(x) = L$ and say the limit of $f(x)$ as x approaches to c from the left (or left hand limit of $f(x)$ as x approaches to c) is equal to L if we can make the values of $f(x)$ arbitrary close to L by taking x to be sufficiently close to c and x less than c. .

If $\lim_{x \to c} f(x) = f(c)$ then we say that $f(x)$ is continuous at $x = c$. A function is continuous on an open interval if it is continuous at each point in the interval. A function is continuous on a closed interval, $[a, b]$, if it is continuous on the open interval (a, b), and

$\lim_{x \to a^+} = f(a)$ **and** $\lim_{x \to b^-} = f(b)$.

Intermediate value theorem:

If f is continuous on the closed interval $[a, b]$, and k is any number between $f(a)$ and $f(b)$ then there exists at least one number c in the interval such that $f(c) = k$. We can use this to help find roots of functions by the bisection method.

The connection between infinite limits and vertical asymptotes:

The line $x = c$ is called a vertical asymptote of a function $f(x)$ if at least on of the following statements is true.

$$\lim_{x \to c} f(x) = \infty \qquad \lim_{x \to c^-} f(x) = \infty \qquad \lim_{x \to c^+} f(x) = \infty$$

$$\lim_{x \to c} f(x) = -\infty \qquad \lim_{x \to c^+} f(x) = -\infty \qquad \lim_{x \to c^-} f(x) = -\infty$$

The connection between limits at infinity and horizontal asymptotes.

The line $y = b$ is called a horizontal asymptote of a function $f(x)$ if either

$$\lim_{x \to \infty} f(x) = b \quad \text{or} \quad \lim_{x \to -\infty} f(x) = b$$

Examples. $\lim_{x \to 0} \frac{1}{x^2} = 0$ $\qquad \lim_{x \to 0^-} \frac{1}{x} = -\infty$ $\qquad \lim_{x \to 0^+} \frac{1}{x} = \infty$

Examle:

$$\lim_{x\to\infty}\frac{-6x^2+7x-11}{3x^2-5x+8}=\lim_{x\to\infty}\frac{x^2(-6+\frac{7}{x}-\frac{11}{x^2})}{x^2(3-\frac{5}{x}+\frac{8}{x^2})}$$

$$=\lim_{x\to\infty}\frac{(-6+\frac{7}{x}-\frac{11}{x^2})}{(3-\frac{5}{x}+\frac{8}{x^2})}=\frac{\lim_{x\to\infty}(-6+\frac{7}{x}-\frac{11}{x^2})}{\lim_{x\to\infty}(3-\frac{5}{x}+\frac{8}{x^2})}$$

$$=\frac{\lim_{x\to\infty}-6+\lim_{x\to\infty}\frac{7}{x}-\lim_{x\to\infty}\frac{11}{x^2}}{\lim_{x\to\infty}3-\lim_{x\to\infty}\frac{5}{x}+\lim_{x\to\infty}\frac{8}{x^2}}=\frac{-6+0-0}{3-0+0}$$

$$=\frac{-6}{3}=-2$$

Properties of limits:

If $\lim_{x\to c}f(x)=\infty$ and $\lim_{x\to c}g(x)=L$, where $0<L<\infty$ then

1. $\lim_{x\to c}\left[f(x)\pm g(x)\right]=\infty$

2. $\lim_{x\to c}\left[f(x)\cdot g(x)\right]=\infty$

3. $\lim_{x\to c}\left[\frac{g(x)}{f(x)}\right]=0$

Derivatives:

The slope of the line tangent to the curve $f(x)$ at $x = c$ can be found by

$f'(c)=\lim_{\Delta x\to 0}\frac{f(c+\Delta x)-f(c)}{\Delta x}=\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$, provided the limit exits.

We have determined the following rules for differentiation:

1. $\frac{d}{dx}[c] = 0$

2. $\frac{d}{dx}[x^n] = nx^{n-1}$

3. $\frac{d}{dx}[cf(x)] = cf'(x)$

4. $\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$

5. $\frac{d}{dx}[\sin x] = \cos x$

6. $\frac{d}{dx}[\cos x] = -\sin x$

7. $\frac{d}{dx}[f(x) \cdot g(x)] = f'(x)g(x) + f(x)g'(x)$

8. $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$

The last property, the quotient rule allows us to prove that

1. $\frac{d}{dx}[\tan x] = \sec^2 x$

2. $\frac{d}{dx}[\cot x] = -\csc^2 x$